Faculty Working Papers

Capacity Expansion of Municipal
Water Treatment Distribution System -
An Application of Dynamic Programming*

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College of Commerce and Business Administration
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February 11, 1972

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*This study is based on a project supported by the Water Resources Center, University of Illinois, Urbana, Illinois.

ABSTRACT

This study investigates the multi-stage capacity expansion of a municipal water treatment system in order to determine the sizes of new treatment plants and the times at which they are added to the system. The capital and operating costs of these plants are given by concave functions reflecting economies of scale available with an increase in capacity. To determine optimum sizes and installation times of the new plants, this expansion problem is formulated to a dynamic programming model.

INTRODUCTION

In recent years, a growing number of studies has shown possibilities of applying mathematical optimization techniques to various water problems. In particular, the dynamic programming method has been proved to be very useful. Examples of the past water studies using this method include time-capacity expansion of urban water systems-comments [Gysi], the determination of aqueduct capacity [Hall, 1963], the design of a multiple purpose reservoire [Hall, 1964], water resource development [Hall and Buras], the optimal sequencing of water supply projects [Butcher, Haimes, and Hall], and multistage water resource systems [Meier and Beighter].

This study is concerned with an economic plan for the multi-stage capacity expansion of a municipal water treatment in order to determine the sizes of water treatment plants and the times at which they are added to the existing system. This system takes care of a particular locality whose demand for water increases with time. The treatment plants of the system are interlocked to one another and assumed to function as an integral unit. Although the system must be designed to satisfy maximum daily demands, considerable part of its capacity is idle much of the time because of seasonal or hourly variations in water use pattern.

The treatment plant and its associated facilities could have a sufficiently large pumping rate to satisfy directly the maximum demand rate of a day, adjusting the pumping rate to a changing demand rate. Such an operation requires a plant capacity that is not fully utilized most of the time. On the other hand, economies in plant operation are usually achieved by running the plant at a constant pumping rate and producing a constant flow of water throughout the operating hours. In this case, excess supply of water during slack periods is stored in distribution

reservoirs and later is used to compensate for the insufficient supply during peak periods or at times of extraordinary demand such as fire fighting.

Economies of scale available with large facilities represent one of the most important aspects in capital investment decision. Normally, the cost of capital investment or operation per unit volume of water treated at capacity decreases with an increase in capacity of a plant. This relationship is usually given by a concave function of capacity. Under increasing demands, a tested and feasible approach that takes advantage of the above economies is to build a sufficiently large plant satisfying demands for some years to come instead of a plant accommodating only the immediate needs.

The investment and operating costs are significantly affected by the source of water. Whether water is obtained from underground sources or surface sources such as reservoirs, rivers, or lakes is determined by the regional condition, quality of available water, or total municipal demand.

Although residential demands for water will exist permanently, the accurate forecasting of these demands in a given community becomes increasingly difficult as the forecast time goes farther into the future mainly because of uncertainties about the population and patterns of living. This study assumes that demands for water in the community under consideration increase each year and can be forecasted over a finite period, and thereafter they will stay constant at the maximum level reached at the end of the period. Therefore, only a finite number of possible capacity increases needs to be considered for each year of the expansion period.

The use of the finite planning period creates analytical difficulties when in reality the system must exist for an indefinite period. To resolve

these difficulties, this study adopts the concept of a permanent chain of identical facilities, a method first proposed by Preinreich [1940]. Thus, a plant added to the system during the expansion period will be succeeded by a permanent chain of plants identical with it.

The objective of the expansion plan is to satisfy given increasing demands over the finite period and to minimize the discounted present value of the capital and operating costs associated with new plants added to the system and the permanent chains of their successors. To determine optimum sizes of the new plants and optimum times at which they are added to the system, the expansion plan is formulated to a dynamic programming model in recursive form suggested by Bellman [1957].

DISCUSSION

Maximum Daily Demands

A long range plan for expanding the capacity of a water treatment-distribution system is preceded by the forecasting of future demand that takes into account past records of the type and pattern of community water use, physical and climatic conditions, expected housing, commercial and industrial developments, and trends of population increase. Significant factors determining water demand in a small residential community include number of residents, number of households, and density of dwelling units.

Normally the design capacity of water treatment is determined by the average annual demand, the maximum daily demand, and the peak hourly demand in a maximum day. The existing FHA standards recommend desinging for an average annual demand of 400 gallons per day (gpd) per dwelling unit, and a peak hourly demand of 2,000 gpd per dwelling unit, except 2,800 gpd per dwelling unit with extensive sprinkling. According to Linaweaver, et.al. (p. 55), however, these standards tend to lead to underdesign of systems in high-valued metered areas and overdesign in lower-valued metered areas and in apartment areas. In place of those standards, they suggest the following formula for determining the expected average demand (p. 58-60):

$$\overline{Q}_t = \{(157 + 3.46V)a + 1.63 \times 10^4 a\overline{L}_s(\overline{E}_{pot} - \overline{P}_{eff})\}\, 10^{-6} \text{ in mgd} \qquad (1)$$

where $\overline{Q}_t$ = expected average demand for the t^{th} year expressed as a rate in million gallons per day

V = average market value in $1,000 per dwelling unit in 1964 prices.

a = number of dwelling units.

$\overline{L}_s$ = average irrigable area in acres per dwelling unit, specifically

$$L_s = 0.803W^{-1.26}$$

W = gross housing density in dwelling units per acre.

$\overline{E}_{pot}$ = estimated average potential evapotranspiration for the period of demand in question in inches of water per day. In the absence of an exact value, $\overline{E}_{pot} = 0.28$ is recommended.

$\overline{P}_{eff}$ = amount of natural precipitation effective in satisfying evapotranspiration for the period and thereby reducing the requirements for lawn sprinkling in inches of water per day.

The expected maximum daily demand $\overline{Q}_{(mxdy)t}$ in the t^{th} year is obtained from (1) by setting $\overline{P}_{eff} = 0$:

$$\overline{Q}_{(mxdy)t} = \{(157 + 3.46V)a + 1.63 \times 10^4 a\overline{L}_s\overline{E}_{pot}\}\ 10^{-6} \text{ in mgd} \qquad (2)$$

In addition to demands created by residential, commercial and industri-- uses, a municipal water system must satisfy requirements for fire-fighting. The American Insurance Association (AIA), which has taken over the former National Board of Fire Underwriters (NBFU), recommends the following flow for the high-value district in an average municipality of 200,000 or less:

$$W_f = 1.020\sqrt{p}\,(1 - 0.01\sqrt{p})10^{-3}$$

where W_f is demand in million gallons per minute and p is population in thousands.

Further, AIA recommends the above fire flow to continue for the number of hours, H_f, specified in Table 1 during a period of 5 days with consumption at the maximum daily rate during any 24-hour period in the past 3 years. When no figure for maximum daily consumption is available its estimate should be at least 50 percent greater than the average daily

Table 1. Required Duration for Fire Flow*

Required Fire Flow W_f gpm	Required Duration H_f hours
Less than 1,250	4
1,250 and greater, but less than 1,500	5
1,500 " " " " " 1,750	6
1,750 " " " " " 2,000	7
2,000 " " " " " 2,250	8
2,250 " " " " " 2,500	9
2,500 and greater	10

*From NBFU Grade Schedule (p. 20, [National Board of Fire Underwriters)].

consumption during the preceding year (p. 14-32, [NBFU]). Assuming the average daily demand increases annually, the AIA recommendations are satisfied by

$$Q_t = \max(\overline{Q}_{(mxdy)t}, 1.5\ \overline{Q}_{t-1}) \qquad \text{in mgd} \qquad (3)$$

$$Q_{(mxdy)t} = Q_t + Q_t^* \qquad \text{in mgd} \qquad (4)$$

$$Q_t^* = 60\ W_f\ H_f \qquad \text{in mgd} \qquad (5)$$

where Q_t and $Q_{(mxdy)t}$ are the maximum daily demand and the design daily requirement, respectively; and Q_t^* is the fire fighting requirement per day.

If the finite expansion period covers T years, various demands defined by (1), (2), and (3) for the years beyond this period stay constant and are given by

$$\begin{aligned} \overline{Q}_t &= \overline{Q}_T \\ \overline{Q}_{(mxdy)t} &= \overline{Q}_{(mxdy)T} \qquad t = T+1,\ T+2,\ \ldots \\ Q_t &= Q_T \end{aligned}$$

Required Treatment Capacities

Design formulas for plant capacity suggested by AIA or various other authors directly satisfy the maximum-day demand and fire-fighting requirements so as to assure a high quality of the treated water at all times. Since such maximum requirements normally last short period of time, in practice the capacity is often determined on a more conservative basis such as the average demand in a peak season. In this case, whenever demands exceed the capacity, the plant output is augmented by booster pumping. We will use this approach in determining the capacity of a new plant.

The system treats water at a constant rate throughout the day. Excess water supplied during the slack period is stored in reservoirs and will be used for equalization during the subsequent peak period. At the beginning the system is assumed to have treatment capacity K_o mgd. The capacity of a plant possibly installed at the beginning of the t^{th} year is represented by K_t, $t = 1, \ldots, T$; K_t is 0 if no plant is added in year t, and, otherwise, it takes a positive value. Each of the installed plants, including the plant existing at the outset, will be replaced by an infinite chain of identical plants at the end of its known economic life.

We now determine constraints imposed on the total capacity of the treatment system as the cumulative sum of the individual plants added to the system. The treatment capacity available in the t^{th} year should satisfy at least the average daily demand $\overline{Q}_t$ by a constant rate operation without booster pumping:

$$\Sigma_{j=0}^{t} K_j \geq \overline{Q}_t \qquad t = 1, \ldots, T \qquad (6)$$

The plants in the system must satisfy the maximum demand Q_t, operated at constant overload rates throughout the day:

$$\Sigma_{j=0}^{t} K_j \geq Q_t/\phi \qquad t = 1, \ldots, T \qquad (7)$$

where $\phi(\geq 1)$ is the coefficient of booster pumping allowable for a prolonged period.

Further, as recommended by AIA, the system should satisfy the fire requirement Q_t^* on a maximum demand day. Assuming that the output rate of

the system can be further boosted from the overload rate satisfying the maximum demand, we write this requirement as follows:

$$(1 - \frac{H_f}{24}) \Sigma_{j=0}^{t} \phi K_j + \frac{H_f}{24} \Sigma_{j=0}^{t} \eta K_j \geq Q_t + Q_t^* \qquad t = 1, \ldots, T \qquad (8)$$

where η is the coefficient of maximum booster pumping used only in periods of extraordinary requirements. On the lefthand of (8), the first term is the volume of water pumped out at a constant rate of overload during the entire day except for the period of fire fighting, and the second term is the volume of water pumped out at the maximum overload rate during the fire. On the righthand, the two terms collectively represent the volume of water required on a maximum demand day with a major fire. This constraint is rewritten to

$$\Sigma_{j=0}^{t} K_j \geq (Q_t + Q_t^*)/\{\phi + \frac{H_f}{24} (\eta-\phi)\} \qquad t = 1, \ldots, T \qquad (9)$$

Constraints (6)-(9) cover all possible situations assumed for the operation of the system. Since the annual cumulative capacity $\Sigma_{j=0}^{t} K_j$ should be at least equal to the righthand of each of Eqs. (6), (7), and (9), the maximum of the values given by these formulas determines the required design capacity of the system in each year and is represented by the following U_t mgd:

$$U_t = \max \begin{bmatrix} \overline{Q}_t \\ Q_t/\phi \\ (Q_t + Q_t^*)/\{\phi + \frac{H_f}{24} (\eta-\phi)\} \end{bmatrix} \qquad t = 1, \ldots, T \qquad (10)$$

A new plant is used to take care of any excess demand over the total capacity of the system existing before its installation. In other words,

the plant installed in year t will satisfy the incremental demand in each year starting with year t over the demand in year t-1 until its capacity is saturated by the incremental demand in some later year, say year s. Previously we used K_t to represent the capacity of a plant possibly installed at the beginning of year t. To replace K_t, we now introduce a more specific symbol, $K_{t,s}$ (s = t, t+1, ..., T), representing the capacity that exactly satisfies the incremental capacity requirement for year s over the requirement for year t-1; this $K_{t,s}$ is given by

$$K_{t,s} = U_s - U_{t-1} \qquad t = 1, \ldots, T \qquad s = t, \ldots, T \tag{11}$$

Total Cost of Capacity Expansion

The total cost of a plant in this problem is composed of its capital cost and annual operating costs. By assumption, each plant installed during the expansion period initiate an infinite chain of plants, identical in size and life, extending beyond the period. Therefore, their capital costs amortized over their lives form a permanent series of identical costs. If $e(K_{t,s})$ represents the capital cost of a plant with capacity $K_{t,s}$, then the sum of the capital costs of all plants in the permanent chain, discounted to the beginning of year t, is given by the following $E(K_{t,s})$:

$$E(K_{t,s}) = (1+R)\, R^{-1}\, ae(K_{t,s}) \qquad t = 1, \ldots, T \qquad s = t, \ldots, T \tag{12}$$

where R is the discount rate, and a is the amortization factor determined jointly by the life of the plant and the discount rate.

The annual operating cost of a plant is composed of the fixed and variable parts. For approximation these costs are determined by the expected average demand $\overline{Q}_t$ and charged at the middle of the year. The variable operating cost is affected by the plant utilization rate. This rate for a plant with capacity $K_{t,s}$ increases annually from year t through year s and thereafter stays constant indefinitely at the maximum value reached in year s. The total operating cost of a permanent chain of plants initiated in year t, discounted to the beginning of that year, is given by the following $F(K_{t,s})$:

$$F(K_{t,s}) = (1+R)^{1/2} R^{-1} f(K_{t,s}) + \Sigma^{s}_{r=t} (1+R)^{-r+t-1/2} g(K_{t,s}, u_r)$$

$$+ (1+R)^{-s + t - 1/2} R^{-1} g(K_{t,s}, u_s) \qquad t = 1, \ldots, T \quad s = t, \ldots, T \qquad (13)$$

where $f(K_{t,s})$ is the annual fixed operating cost of a plant with capacity $K_{t,s}$, and $g(K_{t,s}, u_r)$ is the annual variable operating cost of this plant operated at the following utilization rate u_r in year r:

$$u_r = (\overline{Q}_r - \overline{Q}_{t-1})/(U_s - U_{t-1}) \qquad r = t, \ldots, s \qquad (14)$$

In Eq. (13), the three terms on the righthand represent from left to right the sum of the annual fixed costs over an infinite period, the sum of the annual variable costs from year t through year s, and the sum of the annual variable costs over the infinite period beyond year s, each discounted to the beginning of year t.

The total cost related to a permanent chain of plants is the sum of $E(K_{t,s})$ and $F(K_{t,s})$ and is denoted by $G(K_{t,s})$:

$$G(K_{t,s}) = E(K_{t,s}) + F(K_{t,s}) \tag{15}$$

Dynamic Programming Formulation

Following Bellman's dynamic programming approach [1957], we write the following recursive relationship between the results of two sequences of decisions starting in year t and year t+1:

$$A(t) = G(K_{t,s}) + (1+R)^{-1} A(t+1) \qquad t = 1, \ldots, T \quad s = t, \ldots, T \tag{16}$$

where A(t) is the discounted sum of the capital and operating costs for plants installed in year t and thereafter, A(t+1) is the same sum for plants installed in year t+1 and thereafter.

Since there is no expansion made beyond year T, we write

$$A(t) = 0 \qquad t = T+1, \ldots$$

Our objective is to determine an optimum capacity $K_{t,s}$ for the plant installed in year t so as to minimize the discounted cost A(t). When an optimum decision is made at each decision point, Eq. (16) is replaced by the following equation with A*(t), A*(t+1), and s* representing optimum A(t), A(t+1), and s:

$$A^*(t) = G(K_{t,s^*}) + (1+R)^{-1} A^*(t+1) \qquad t = 1, \ldots, T \tag{17}$$

Using Eq. (17), we start in year T and determine optimum K_{T,s_T^*} where s_T^* is an optimum s for t = T, then determine $K_{T-1,\, s_{T-1}^*}$ for year T-1, and so on, working backward and determining an optimum capacity for each year. When we compute A*(1) and determine $K_{1,\, s_1^*}$, we have finally obtained

a complete optimum solution to the problem. This backward process is explained step by step by the following equations:

$$A^*(T) = G(K_{T,s^*_T})$$

$$A^*(T-1) = G(K_{T-1,s^*_{T-1}}) + (1+R)^{-1} A^*(T)$$

$$\cdots$$

$$A^*(1) = G(K_{1,s^*_1}) + (1+R)^{-1} A^*(2)$$

There are two possible cases regarding the existing treatment capacity at each evaluation point; (1) the capacity satisfies the requirement and a new plant is not needed until year r, and (2) the capacity is insufficient for the requirement and a new plant is to be installed at this point. In the first case, Eq. (17) can be simplified to

$$A^*(t) = (1+R)^{-r+t} A^*(r) \qquad \begin{matrix} t = 1, \ldots, T \\ r = t+1, \ldots, T \end{matrix} \tag{18}$$

In the second case, the capacity K_{t,s^*} of the new plant makes it unnecessary to install another plant before year s^*+1, thus making years $t+1, \ldots, s^*-1, s^*$ belong to the first case given by Eq. (18). Therefore, in this case Eq. (17) may be replaced by

$$A^*(t) = G(K_{t,s^*}) + (1+R)^{-(s^* + 1-t)} A^*(s^*+1) \qquad t = 1, \ldots, T \tag{19}$$

Since s^* in (19) is selected from the alternative values of s ($s = t, t+1, \ldots,$ or T) so as to minimize $A(t)$, Eq. (19) may be rewritten to

$$A^*(t) = \min_s \begin{bmatrix} s = t: & G(K_{t,t}) + (1+R)^{-1} A^*(t+1) \\ s = t+1: & G(K_{t,t+1}) + (1+R)^{-2} A^*(t+2) \\ \cdots & \\ s = T-1: & G(K_{t,T-1}) + (1+R)^{-T+t} A^*(T) \\ s = T: & G(K_{t,T}) \end{bmatrix} \qquad t = 1, \ldots, T \tag{20}$$

Thus, with Eqs. (15), (18) and (20), we have completed the dynamic programming formulation of the capacity expansion problem.

Cost Functions of Water Treatment

Specific cost functions for $E(K)$ and $F(K)$ used in (15) are required for the subsequent numerical example showing an application of the dynamic programming model that we have developed. However, our discussion on the cost functions must be based on only a few studies available on the subject.

The most extensive study on the costs of surface-water treatment was conducted by Koenig [1963]. In his study, the capital cost of a plant covers the low lift pumping station, the treatment plant itself and the high lift pumping station, but it does not include conveyance lines for raw water or finished water, nor booster stations on finished water lines or distribution lines (p. 295, [Koenig]). The Illinois State Water Survey (ISWS) [1968] adjusted data from 42 plants (including Koenig's 30 plants and other data which appeared in the Journal of the American Water Works Association) to 1964 prices and obtained the following regression relation between capacity and capital cost for surface water treatment:

$$E_{(s)}(K) = 267.9K^{.65} \qquad \text{in } \$1000$$

where $E_{(s)}(K)$ is the capital cost in \$1000 and K is the capacity in mgd.

For ground water treatment, ISWS obtained the following formula for capital cost from data on 58 plants located in the State of Illinois:

$$E_{(g)}(K) = 115\ K^{.63} \qquad \text{in } \$1000$$

Using Koenig's data in 1964 prices, we obtained unit and total daily costs of surface water treatment (p. 5, [Hinomoto]). From these costs, the annual fixed and variable operating costs for capacity K at utilization rate u are obtained and given by the following $f_{(s)}(K)$ and $g_{(s)}(K, u)$, respectively:

$$f_{(s)}(K) = 1.121\ K^{.481} + 9.964\ K^{.687} + .372\ K^{.930} \quad \text{in } \$1000/\text{Yr.}$$

$$g_{(s)}(K, u) = u\ (4.380\ K^{.764} + 10.147\ K^{.718}) + u^{.5}\ (14.782\ K^{.579}) \quad \text{in } \$1000/\text{Yr.}$$

A study reported by ISWS (p. 2, [ISWS]) indicates ground water treatment costs are in general approximately 43% of the same costs for surface water treatment. Therefore, we write the fixed and variable operating costs for a ground water treatment plant, denoted by $f_{(g)}(K)$ and $g_{(g)}(K, u)$, as follows:

$$f_{(g)}(K) = .43\ f_{(s)}(K)$$
$$g_{(g)}(K, u) = .43\ g_{(s)}(K, u)$$

Numerical Example

This example approximates the conditions existing in the Champaign-Urbana twin-city area, Illinois. The expansion plan covers the period from 1970 through 1985, regarding 1970 as year 1.

As of the beginning of year 1 the existing water treatment system serving the community is composed of two treatment plants, each with a capacity of 9 mgd; thus K_o = 18 mgd. This system needs a long-range plan for its capacity expansion to cope with continuously growing demand over a period of 16 years beyond which demand is expected to stay constant at the level

attained in the last year of the period. Water is obtained from underground sources which can accommodate the requirements of the area for an indefinite future. The demands have been forecast through Eqs. (1)-(4) using estimated values of factors such as population, average market value of a dwelling unit, number of people in dwelling unit, number of dwelling units, and residential area in the region. The forecast demands over the 16-year period are listed in Table 2 where the maximum daily requirement Q_t is the larger of 1.5 times the expected demand $\overline{Q}_{t-1}$ obtained from (1) or the expected maximum demand $\overline{Q}_{(mxdy)t}$ obtained from (2). Then the capacity requirement for each year is determined by Eq. (10) and listed in the right-most column of Table 2.

The duration and total demand of the peak period on a maximum demand day, expressed as fractions of the duration and demand of the entire day, are assumed to be $\alpha = .6$ and $\beta = .91$, respectively. The life of a plant is assumed to be 30 years, and the annual rate of interest on bonds issued for plant construction is 8%; the amortization factor of the plant determined by these values is $a = .0888$. Further the annual discount rate, or the yield rate expected of the system by the agency, in cost formulas (12) and (13) is $R = 10\%$. Two types of booster pumping have been considered; one, that is used over a prolonged period on maximum demand days, is represented by $\phi = 1.20$, and the other, that is used in short periods of extraordinary demands such as fire-fighting on a maximum demand day, is represented by $\eta = 1.30$.

Before determining an optimum expansion of the treatment capacity using Eq. (20), it is necessary to obtain the capital and operating costs of a plant possibly installed in each year. The plant to be installed in year t

Table 2. Estimated Water Requirements

Year t	Expected Average Demand $\overline{Q}_t$ mgd	1.5 times $\overline{Q}_t$ $1.5\ \overline{Q}$ mgd	Expected Maximum Demand $\overline{Q}_{(mxdy)t}$ mgd	Design Maximum Demand[1] Q_t mgd	Fire Fighting Requirement Q_t^* mg	Capacity Requirement[2] U_t mgd
						(18.00)
1	14.42	21.63	20.59	20.84	5.30	21.69
2	14.97	22.46	21.48	21.63	5.33	22.38
3	15.54	23.31	22.40	22.46	5.37	23.10
4	16.12	24.18	23.32	23.31	5.40	23.82
5	16.71	25.07	24.25	24.25	5.44	24.57
6	17.33	26.00	25.21	25.21	5.49	25.36
7	17.95	26.93	26.18	26.18	5.53	26.14
8	18.63	27.95	27.20	27.20	5.59	27.01
9	19.31	28.97	28.21	28.21	5.64	27.87
10	20.01	30.02	29.24	29.24	5.70	28.77
11	20.74	31.11	30.30	30.30	5.77	29.70
12	21.51	32.27	31.39	31.39	5.84	30.69
13	22.30	33.45	32.50	32.50	5.91	31.70
14	23.14	34.71	33.64	33.64	5.99	32.78
15	24.02	36.03	34.82	34.82	6.08	33.91
16	24.95	37.43	36.04	36.04	6.18	35.12

[1] The value represents the larger of $1.5\ \overline{Q}_{t-1}$ or $\overline{Q}_{(mxdy)t}$.

[2] Based on booster pumpage with $\phi = 1.20$ and $\eta = 1.30$.

could have one of alternative capacities, denoted by $K_{t,s}$, which equal the incremental capacity requirements for year s (s = t, ..., T) over the total requirement for year t-1. The values of $K_{t,s}$ for t = 1, ..., T are listed in Table 3, computed through Eq. (11) using capacity requirement U_t listed in Table 2. Then, with $K_{t,s}$ thus obtained, the total costs $G(K_{t,s})$ are determined through Eq. (15) and listed in Table 4.

We now determine an optimum policy for expansion in each year of the expansion period, starting in year 1985 and working backward year by year to year 1. For year 16, there is only one capacity to be considered, i.e., $K_{16,\ 16}$ or 1.21 mgd as listed in Table 3. In Table 4, the total cost of a permanent chain of plants with capacity $K_{16,\ 16}$ = 1.21 mgd is found to be \$297,260. Thus, the cost of the optimum expansion plan initiated in year 16 is given by

$$A^*(16) = G(K_{16,\ 16}) = \$297,260$$

For year 15, two alternative capacities are to be evaluated; one capacity is $K_{15,\ 15}$ that satisfies the requirement for year 15 only, and the other is $K_{15,\ 16}$ that satisfies the requirements for year 15 and year 16.

The values of $K_{15,\ 15}$ and $K_{15,\ 16}$ are listed in Table 3 as 1.14 mgd and 2.34 mgd, respectively. Further from Table 4, we find the total costs $G(K_{15,\ 15})$ and $G(K_{15,\ 16})$ to be \$286,120 and \$467,420. The expansion plan for installing a plant with capacity $K_{15,\ 15}$ or $K_{15,\ 16}$ in year 15 gives the following discounted total cost:

$$S = 15: \quad G(K_{15,\ 15}) + (1+R)^{-1} A^* (16)$$

$$= 286,120 + 270,240 = \$556,360$$

$$S = 16: \quad G(K_{15,\ 16}) = \$467,420$$

Table 3

$K_{t,s}$ mgd: Capacity Requirement in Year s for a Plant Installed in Year t.

t	s: 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3.69	4.38	5.10	5.82	6.57	7.36	8.14	9.01	9.87	10.77	11.70	12.69	13.70	14.78	15.91	17.12
2		0.69	1.41	2.13	2.88	3.67	4.45	5.32	6.19	7.08	8.01	9.00	10.01	11.09	12.23	13.43
3			0.72	1.44	2.19	2.98	3.76	4.63	5.49	6.39	7.32	8.31	9.32	10.40	11.53	12.74
4				0.72	1.47	2.26	3.04	3.91	4.78	5.67	6.60	7.59	8.60	9.68	10.82	12.02
5					0.75	1.54	2.32	3.19	4.05	4.94	5.88	6.87	7.88	8.96	10.09	11.30
6						0.79	1.57	2.44	3.30	4.20	5.13	6.12	7.13	8.21	9.34	10.55
7							0.78	1.65	2.51	3.41	4.34	5.33	6.34	7.42	8.55	9.76
8								0.87	1.73	2.63	3.56	4.55	5.56	6.64	7.77	8.98
9									0.86	1.76	2.69	3.68	4.69	5.77	6.90	8.11
10										0.89	1.83	2.82	3.83	4.90	6.04	7.25
11											0.93	1.92	2.93	4.01	5.15	6.35
12												0.99	2.00	3.08	4.21	5.42
13													1.01	2.09	3.22	4.43
14														1.08	2.21	3.42
15															1.14	2.34
16																1.21

Table 4

$G(K_{t,s})$, Discounted Total Cost of a Permanent Chain of Plants with Capacity $K_{t,s}$ Initiated in Year t (in $1000)

	s															
t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	589.19	682.77	753.46	820.93	886.84	952.71	1014.78	1080.48	1142.58	1204.07	1265.51	1327.82	1388.53	1450.90	1513.83	1578.05
2		209.31	338.99	437.46	523.83	603.96	675.57	747.82	813.85	877.28	939.15	1000.66	1059.74	1119.61	1179.38	1239.88
3			214.08	344.43	444.66	534.67	613.53	691.99	762.95	830.56	896.07	960.81	1022.73	1085.18	1147.29	1209.94
4				216.00	348.90	453.45	542.11	628.52	705.60	778.26	848.09	916.64	981.83	1047.25	1112.04	1177.14
5					219.66	357.65	460.49	557.45	642.23	721.06	796.04	869.04	938.01	1006.84	1074.67	1142.52
6						227.04	363.03	475.34	570.39	657.03	738.34	816.68	890.11	962.90	1034.22	1105.21
7							226.18	374.41	484.65	581.92	671.39	756.40	835.27	912.80	988.25	1062.90
8								241.42	386.15	498.40	598.48	691.73	777.12	860.20	940.40	1019.24
9									240.59	389.63	506.04	610.91	705.00	795.25	881.42	965.41
10										245.84	399.37	520.78	626.24	725.38	818.74	908.77
11											252.79	412.46	534.44	645.46	747.93	845.38
12												262.16	422.84	551.22	665.92	772.83
13													265.61	434.85	567.64	687.37
14														277.19	451.34	589.43
15															286.12	467.42
16																297.26

Therefore, the optimum expansion plan for year 15 and forward is

$$A^*(15) = G(K_{15,\ 16}) = \$467,420$$

For year 14, three alternative capacities need to be evaluated: $K_{14,\ 14}$ satisfying the requirement for year 14 only, $K_{14,\ 15}$ satisfying the requirements for year 14 and year 15, and $K_{14,\ 16}$ satisfying the requirements for years 14, 15, and 16. These capacities give the following total costs:

$$S = 14: \quad G(K_{14,\ 14}) + (1+R)^{-1}\ A^*(15)$$
$$= 277,190 + 424,930 = \$702,120$$
$$S = 15: \quad G(K_{14,\ 15}) + (1+R)^{-2}\ A^*(16)$$
$$= 451,340 + 245,670 = \$697,010$$
$$S = 16: \quad G(K_{14,\ 16}) = \$589,430$$

Among the three capacities, capacity $K_{14,\ 16}$ gives the minimum total cost. Therefore, the optimum expansion plan initiating in year 14 is

$$A^*(14) = G(K_{14,\ 16}) = \$589,430$$

Similarly, we obtain the optimum expansion plan initiating each of years 13, 12, ..., and 1. As in the case of year 16, 15, or 14, all of the optimum plans for years 13, ..., 3 install one plant that will take care of the incremental requirements for all remaining years of the expansion period.

However, each of the optimum plans for year 2 and year 1 involves two plants and gives the following total cost:

$$A^*(2) = G(K_{2,\,10}) + (1+R)^{-9}\, A^*(11)$$

$$= 877{,}280 + 358{,}520 = \$1{,}235{,}800$$

$$A^*(1) = G(K_{1,\,9}) + (1+R)^{-9}\, A^*(10)$$

$$= 1{,}142{,}580 + 385{,}410 = \$1{,}527{,}990$$

The optimum plan for year 1 is the final solution to the present dynamic programming model, requiring a total cost of $1,527,990. It states:

(1) To install in year 1 a plant with the capacity $K_{1,\,9}$ that, in combination with the existing plants, satisfies the requirements for years 1, ..., 9. This capacity is

$$K_{1,\,9} = U_9 - U_0 \ (= K_o = 18 \text{ mgd})$$
$$= 9.87 \text{ mgd}$$

(2) To install in year 10 a plant with the capacity $K_{10,\,16}$ that in combination with the existing plants satisfies the requirements through year 16. This capacity is

$$K_{10,\,16} = U_{16} - U_9$$
$$= 7.25 \text{ mgd}$$

Table 5 lists the optimum expansion plan obtained for each of years 16, ..., 1 and the total cost discounted to the beginning of that year.

Table 5. Dynamic Programming Solutions to Optimum Capacity Expansion Plan Initiated in Each Year of the Expansion Period.

Year t	Optimum Expansion Plan Starting in year t Expressed in Cost Formula $A^*(t)$	Discounted Total Cost of Optimum Expansion Plan[1]
16	$G(K_{16,\ 16})$	\$ 297,260
15	$G(K_{15,\ 16})$	\$ 467,420
14	$G(K_{14,\ 16})$	\$ 589,430
13	$G(K_{13,\ 16})$	\$ 687,370
12	$G(K_{12,\ 16})$	\$ 772,830
11	$G(K_{11,\ 16})$	\$ 845,380
10	$G(K_{10,\ 16})$	\$ 908,770
9	$G(K_{9,\ 16})$	\$ 965,410
8	$G(K_{8,\ 16})$	\$ 1,019,240
7	$G(K_{7,\ 16})$	\$ 1,062,900
6	$G(K_{6,\ 16})$	\$ 1,105,210
5	$G(K_{5,\ 16})$	\$ 1,142,520
4	$G(K_{4,\ 16})$	\$ 1,177,140
3	$G(K_{3,\ 16})$	\$ 1,209,940
2	$G(K_{2,\ 10}) + (1+R)^{-9} A^*(11)$	\$ 1,235,800
1	$G(K_{1,\ 9}) + (1+R)^{-9} A^*(10)$	\$ 1,527,990

[1]The value is discounted to the beginning of year t.

SUMMARY

This study has formulated a model representing the capacity expansion of an existing water-treatment system serving a residential community. It assumes demands increase with time and can be forecast for certain over a finite period beyond which they stay constant indefinitely at the maximum levels attained at the end of the period. To satisfy those demands with a minimum cost, new treatment plants of optimum sizes are added to the system at proper points in time during the period. Each of the new plants is replaced by a permanent chain of plants identical with it in capacity and economic life so as to satisfy the constant demands beyond the expansion period.

Two important factors determining the capacity of a municipal water-treatment system are the expected maximum daily demand mainly influenced by lawn sprinkling on hot summer days and the fire fighting requirements recommended by the American Insurance Association. The design capacities determined by the formulas suggested by various authors tend to be much greater than the capacity used in practice. To resolve the discrepancies between these formulas and the actual practice, we have introduced two coefficients of booster pumping; one coefficient represents a rate of overload on the design capacity used throughout a maximum demand day, and the other represents the maximum rate of overload applicable to fire fighting during the peak-period of that day.

The objective of the capacity expansion problem is to minimize the total cost of investment and operation associated with new plants and their permanent chains of successors. The capital and annual operating costs of a plant are given by exponential functions of capacity in concave form reflecting economies of scale associated with a larger capacity.

Empirical cost functions in this form are used in the subsequent numerical example. The system is to satisfy the annual capacity requirement determined by the maximum of the following three treatment requirements: the first requirement is for an average demand day at a constant rate without booster pumping, the second is for a maximum demand day at a constant rate with booster pumping throughout the day, and the third is for fire fighting lasting a duration specified by AIA during the peak-period on a maximum demand day at a maximum rate of booster pumping.

A new plant is added to the system whenever the existing capacity becomes short of the annual capacity requirement determined above. To determine optimum capacities and installation times of the new plants, the expansion plan is formulated to a dynamic programming model. This model is applied to a case approximating the conditions in the Champaign-Urbana area over the period 1970-1985.

Most capital investment decisions are based on trade-offs between the cost of over-capacity and the penalty of under-capacity for given requirements. In the present problem, over-capacity is essnetially an economic matter involving the supplier of water alone and its cost could be determined with some accuracy. On the other hand, the effects of under-capacity would fall on the consumer of water, rather than the supplier, in the forms of higher fire insurance rates, shortage of water in the home and community, or poorly treated water. The involvement of non-economic factors makes it difficult to ascertain the penalty of under-capacity. The absence of the effects of under-capacity in the present model, however, is not intended to minimize their significance.

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